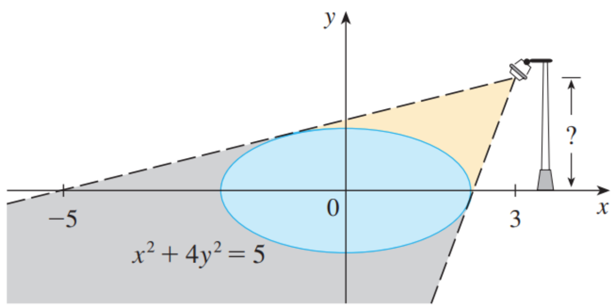
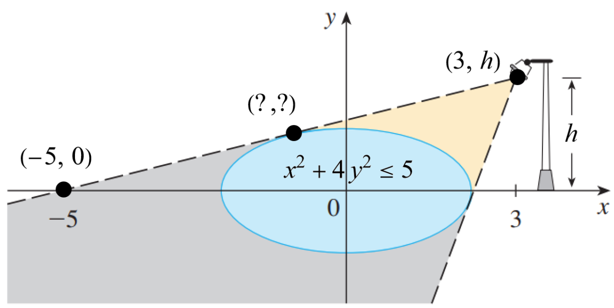
Exercise 80

The figure shows a lamp located three units to the right of the y-axis and a shadow created by the elliptical region $x^2 + 4y^2 \leq 5$. If the point $(-5, 0)$ is on the edge of the shadow, how far above the x-axis is the lamp located?



Solution

Let the height of the lamp be h.



Calculate the slope of the upper tangent line.

$$m = \frac{h - 0}{3 - (-5)} = \frac{h}{8}$$

Use the point-slope formula to get an equation for this line.

$$y - 0 = \frac{h}{8}(x - (-5))$$

$$y = \frac{h}{8}(x + 5)$$

Differentiate both sides of the ellipse's equation with respect to x.

$$\frac{d}{dx}(x^2 + 4y^2) = \frac{d}{dx}(5)$$

Use the chain rule to differentiate $y = y(x)$.

$$2x + 8y\frac{dy}{dx} = 0$$

Solve for dy/dx.

$$8y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{4y}$$

At the upper point of intersection (x_0, y_0), the slope of the tangent line is

$$\frac{h}{8} = -\frac{x_0}{4y_0}.$$

Multiply both sides by 8.

$$h = -\frac{2x_0}{y_0} \tag{1}$$

To find the upper point of intersection between the tangent line and the ellipse, solve the following system of equations.

$$\begin{cases} y_0 = \dfrac{h}{8}(x_0 + 5) \\ \\ x_0^2 + 4y_0^2 = 5 \end{cases}$$

Substitute the formula for y_0 into the second equation.

$$x_0^2 + 4\left[\frac{h^2}{64}(x_0+5)^2\right] = 5$$

$$x_0^2 + \frac{h^2}{16}(x_0+5)^2 = 5$$

$$16x_0^2 + h^2(x_0+5)^2 = 80$$

$$16x_0^2 + h^2(x_0^2 + 10x_0 + 25) = 80$$

$$(16+h^2)x_0^2 + 10h^2x_0 + 25h^2 = 80$$

$$(16+h^2)x_0^2 + 10h^2x_0 + 5(5h^2 - 16) = 0$$

Solve for x_0 using the quadratic formula.

$$x_0 = \frac{-10h^2 \pm \sqrt{100h^4 - 4(16+h^2)5(5h^2-16)}}{2(16+h^2)}$$

$$x_0 = \frac{-10h^2 \pm \sqrt{5120 - 1280h^2}}{2(16+h^2)}$$

$$x_0 = \frac{-10h^2 \pm 16\sqrt{5(4-h^2)}}{2(16+h^2)}$$

As a result,

$$x_0 = \frac{-5h^2 \pm 8\sqrt{5(4-h^2)}}{16+h^2}.$$

For the intersection of the ellipse with the upper tangent line in particular, choose the negative sign so that x_0 is negative.

$$x_0 = \frac{-5h^2 - 8\sqrt{5(4-h^2)}}{16+h^2}$$

The value of y_0 corresponding to this value of x_0 is

$$\begin{aligned} y_0 &= \frac{h}{8}(x_0+5) \\ &= \frac{h}{8}\left[\frac{-5h^2 - 8\sqrt{5(4-h^2)}}{16+h^2} + 5\right] \\ &= \frac{h}{8}\left[\frac{80 - 8\sqrt{5(4-h^2)}}{16+h^2}\right] \\ &= \frac{10h - h\sqrt{5(4-h^2)}}{16+h^2}. \end{aligned}$$

Now substitute this pair of values for x_0 and y_0 into equation (1).

$$h = -\frac{2\left[\frac{-5h^2-8\sqrt{5(4-h^2)}}{16+h^2}\right]}{\frac{10h-h\sqrt{5(4-h^2)}}{16+h^2}} = \frac{10h^2 + 16\sqrt{5(4-h^2)}}{10h - h\sqrt{5(4-h^2)}}$$

Solve for h.

$$\begin{aligned} h\left[10h - h\sqrt{5(4-h^2)}\right] &= 10h^2 + 16\sqrt{5(4-h^2)} \\ \cancel{10h^2} - h^2\sqrt{5(4-h^2)} &= \cancel{10h^2} + 16\sqrt{5(4-h^2)} \\ -h^2\sqrt{5(4-h^2)} &= 16\sqrt{5(4-h^2)} \\ -h^2\sqrt{5(4-h^2)} - 16\sqrt{5(4-h^2)} &= 0 \\ -(h^2+16)\sqrt{5(4-h^2)} &= 0 \\ \sqrt{5(4-h^2)} &= 0 \\ 5(4-h^2) &= 0 \\ 4-h^2 &= 0 \\ h &= 2 \end{aligned}$$